Simplified Eulerian-Frame Material-Motion Corrections for Radiative Transfer Calculations

J. E. Morel
Transport Methods Group, XTM
Applied theoretical and Computational Physics Division
Los Alamos National Laboratory
Los Alamos, NM 87545

Abstract

Simplified material-motion corrections are presented for the Eulerian-frame radiative transfer equation in a regime characterized by material velocities that are very small relative to the speed of light. Our corrections are intended to accurately treat only the overall exchange of energy and momentum between the radiation and material fields. Detailed radiation spectral effects are not accurately treated by our corrections. Correction terms that are fully accurate to $O(v/c)$ have a differential form that makes them costly and difficult to implement. Our simplified corrections have a non-differential form, but they result in total energy conservation and momentum conservation, and they are correct to $O(v/c)$ in the equilibrium diffusion limit.

The purpose of this report is to describe a simplified material-motion treatment for the Eulerian-frame radiative transfer equation. In the Eulerian frame, material motion effectively modifies the photon interaction cross-sections and sources in a fairly complex way. Incorporating the exact modifications leads to equations that are very difficult to solve numerically. Neglecting these corrections leads to a lack of energy conservation. In particular, the kinetic energy change in the material due to radiation momentum deposition is not removed from the radiation field. Even if the ratio of material speed to the speed of light (v/c) is small, a significant error can occur over time due to the steady accumulation of small conservation errors.

We restrict our attention to a regime characterized by $v^2/c^2 \ll v/c$. Material-motion corrections must be made to the Eulerian-frame radiation transport equation in this regime, but the non-relativistic hydrodynamic equations need not be modified because they are correct to $O(v/c)$.

There are two approaches for performing radiation transport calculations in this regime:

1. Use the Eulerian-frame transport equation in conjunction with opacities and sources expanded to first-order in v/c.

2. Use the co-moving or fluid-frame transport equation expanded to first-order in v/c.

The first approach results in a transport equation that contains frequency derivatives of

the absorption cross-section together with frequency derivatives of certain moments of the radiation intensity. The second approach results in a transport equation that contains angular and frequency derivatives of the radiation intensity. The absorption cross-section often varies rapidly with frequency due to resonance effects. Such behavior in the absorption cross-section causes similar behavior in the radiation intensity. Thus frequency derivatives of the absorption cross section and frequency derivatives of the radiation intensity are difficult to treat numerically.

It is our purpose to describe a simplified material-motion treatment that avoids the difficulties associated with traditional approaches. A simplified treatment is possible because we seek only to obtain accurate radiation energy and momentum deposition in the transport material. In principle, this makes it possible to obtain accurate hydrodynamics solutions even though detailed differential effects of material motion on the radiation intensity are neglected.

Our material motion treatment has several key properties:

1. Energy and momentum are conserved.

2. The treatment is exact in the streaming (optically-thin) limit, and correct to $O(v/c)$ in the equilibrium diffusion limit.

3. The treatment is exact in a certain sense for an infinite medium in equilibrium moving

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at a constant speed.

We begin the description of our method with the Eulerian-frame radiation transport equation:

$$\frac{1}{c}\frac{\partial I}{\partial t} + \overrightarrow{\Omega} \cdot \overrightarrow{\nabla} \cdot I + \sigma_t I = \frac{1}{4\pi}\sigma_s c E_\nu + \sigma_a B(T) \quad , \tag{1}$$

where

t is the time,

$\overrightarrow{\Omega}$ is the angular variable,

ν is the frequency,

T is the material temperature,

σ_a is the absorption cross section $(length^{-1})$,

σ_s is the scattering cross section $(length^{-1})$,

σ_t is the total cross-section $(\sigma_a + \sigma_s)$,

I is the angular intensity $(energy/area - time - frequency - steradian)$,

E_ν is the frequency-dependent radiation energy density $(energy/volume - frequency)$,

and B is the Planck function $(energy/area - time - frequency - steradian)$.

Note that cE_ν represents the angle-integrated intensity:

$$cE_\nu = \int_{4\pi} I\, d\Omega \quad , \tag{2}$$

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and that

$$B = \frac{2h\nu^3}{c^2} \left[\exp\left(h\nu/kT\right) - 1 \right]^{-1} \quad , \tag{3}$$

where h is Planck's constant and k is Boltzmann's constant.

The central theme of our approximation is to add non-differential terms to Eq. (1) which ensure that the radiation energy equation and the radiation momentum equation are correct to $O(v/c)$ in the equilibrium diffusion limit. Specifically, the corrected version of Eq. (1) is:

$$\frac{1}{c}\frac{\partial I}{\partial t} + \overrightarrow{\Omega} \cdot \overrightarrow{\nabla} \cdot I + \sigma_t I = \frac{1}{4\pi}\sigma_s c E_\nu + \sigma_a B(T) + \frac{1}{4\pi}C_0 + \frac{3}{4\pi}\overrightarrow{C}_1 \cdot \overrightarrow{\Omega} \quad , \tag{4}$$

where the coefficients of the correction terms are:

$$C_0 = \left(\sigma_a - \sigma_s\right) \overrightarrow{F}_\nu^* \cdot \frac{\overrightarrow{v}}{c} \quad , \tag{5}$$

$$\overrightarrow{C}_1 = \sigma_t \frac{4}{3} c E_\nu \frac{\overrightarrow{v}}{c} \quad , \tag{6}$$

where $\overrightarrow{F}_\nu^*$ is an approximation to the fluid-frame frequency-dependent radiation flux:

$$\overrightarrow{F}_\nu^* = \overrightarrow{F}_\nu - \frac{4}{3} c E_\nu \frac{\overrightarrow{v}}{c} \quad , \tag{7}$$

and $\overrightarrow{F}_\nu$ is the frequency-dependent radiation flux:

$$\overrightarrow{F}_\nu = \int_{4\pi} \overrightarrow{\Omega}\, I\, d\Omega \quad . \tag{8}$$

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In order to physically understand what the correction term is doing, we first integrate Eq. (4) over all angles and frequencies to obtain the corrected radiation energy equation:

$$\frac{\partial E_r}{\partial t} + \vec{\nabla} \cdot \vec{F} + \int_0^\infty \sigma_a \left(cE_\nu^* - 4\pi B \right) \, d\nu = -\rho \, \vec{a}_r \cdot \vec{v} \quad , \tag{9}$$

where E_r is the total radiation energy density:

$$E_r = \frac{1}{c} \int_0^\infty \int_{4\pi} I \, d\Omega \, d\nu \quad , \tag{10}$$

$\vec{F}$ is the total radiation energy flux:

$$\vec{F} = \int_0^\infty \int_{4\pi} \vec{\Omega} \, I \, d\Omega \, d\nu \quad , \tag{11}$$

cE_ν^* is an approximation to the fluid-frame angle-integrated intensity:

$$cE_\nu^* = cE_\nu - 2 \vec{F}_\nu^* \cdot \frac{\vec{v}}{c} \quad , \tag{12}$$

and $\vec{a}_r$ is an approximation to the radiation acceleration (i.e., the material acceleration due to radiation momentum deposition:

$$\vec{a}_r = \int_0^\infty \frac{\sigma_t}{\rho c} \vec{F}_\nu^* \, d\nu \quad . \tag{13}$$

Next we multiply Eq. (4) by $\frac{1}{c} \vec{\Omega}$ and integrate over all angles and frequencies to obtain the corrected radiation momentum equation:

$$\frac{1}{c^2} \frac{\partial \vec{F}}{\partial t} + \vec{\nabla} \cdot \vec{P} + \int_0^\infty \frac{\sigma_t}{c} \vec{F}^* \, d\nu = 0 \quad , \tag{14}$$

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where $\overrightarrow{\overrightarrow{P}}$ is the radiation pressure tensor:

$$\overrightarrow{\overrightarrow{P}} = \frac{1}{c} \int_{4\pi} \int_0^\infty \overrightarrow{\Omega}\,\overrightarrow{\Omega}\, I \, d\nu \, d\Omega \quad . \tag{15}$$

It is appropriate at this point to discuss the origins of our expressions for $\overrightarrow{F}^*_\nu$ and cE^*_ν. An expression for the fluid-frame total radiation flux that is correct to $O(v/c)$ is [1]:

$$\overrightarrow{F}^* = \overrightarrow{F} - cE_r \frac{\overrightarrow{v}}{c} - c\overrightarrow{\overrightarrow{P}} \cdot \frac{\overrightarrow{v}}{c} \quad , \tag{16}$$

To obtain Eq. (7), we first substitute $\frac{1}{3}\overrightarrow{\overrightarrow{E}}_r$ for $\overrightarrow{\overrightarrow{P}}$ in Eq. (16):

$$\overrightarrow{F}^* = \overrightarrow{F} - \frac{4}{3}cE_r \frac{\overrightarrow{v}}{c} \quad , \tag{17}$$

This substitution is exact in the equilibrium diffusion limit. Next we obtain Eq. (7) by simply replacing all the quantities in Eq. (17) with their frequency-dependent counterparts. There is really no justification for this other than the fact that this substitution yields the properties we seek in our corrected transport equation. An expression for the fluid-frame angle/frequency-integrated radiation intensity that is correct to $O(v/c)$ is [1]:

$$cE^*_r = cE_r - 2\overrightarrow{F} \cdot \frac{\overrightarrow{v}}{c} \quad , \tag{18}$$

To obtain Eq. (12), we first substitute $\overrightarrow{F}^*$ (as defined in Eq. (17)) for $\overrightarrow{F}$ in Eq. (18):

$$cE^*_r = cE_r - 2\overrightarrow{F}^* \cdot \frac{\overrightarrow{v}}{c} \quad , \tag{19}$$

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It can be seen from Eq. (17) that $\overrightarrow{F}^*$ differs from $\overrightarrow{F}$ by a term of $O(v/c)$. Since $\overrightarrow{F}$ in Eq. (18) is multiplied by a term of $O(v/c)$, the substitution of $\overrightarrow{F}^*$ for $\overrightarrow{F}$ results in a change of $O(v^2/c^2)$. Thus Eq. (19) is correct to $O(v/c)$. Next we obtain Eq. (12) from Eq. (19) simply by replacing all the quantities in Eq. (19) with their frequency-dependent counterparts. Again, there is really no justification for this other than the fact that this substitution yields the properties we seek in our corrected transport equation.

Examining Eqs. (9) and (14), and recognizing that $\rho\,\overrightarrow{a}_r\cdot\overrightarrow{v}$ represents the rate of change per unit volume of the material kinetic energy due to radiation momentum deposition, we see that the correction term in Eq. (4) achieves the following:

1. The uncorrected Eulerian-frame total energy absorption rate is replaced with an approximate fluid-frame total energy absorption rate.

2. The uncorrected Eulerian-frame total momentum deposition rate is replaced with an approximate fluid-frame total momentum deposition rate.

3. The rate of change per unit volume in material kinetic energy due to radiation momentum deposition is subtracted from the radiation energy field.

We identify the photon energy deposited in the material internal energy field and the photon momentum deposited in the material with the corrected reaction rates. For instance, in

a Lagrangian calculation, the internal material energy equation would take the following form:

$$\frac{De}{Dt} + P_m \overrightarrow{\nabla} \cdot \overrightarrow{v} = \int_0^\infty \sigma_a \left(cE_v^* - 4\pi B \right) d\nu \quad , \tag{20}$$

where P_m is the material pressure. Similarly, the momentum equation would be:

$$\frac{D\overrightarrow{v}}{Dt} = -\frac{1}{\rho} \overrightarrow{\nabla} P_m + \overrightarrow{a}_r \quad . \tag{21}$$

Taking the dot product of the material velocity with Eq. (21), we obtain the kinetic energy equation:

$$\frac{1}{2}\frac{D\overrightarrow{v} \cdot \overrightarrow{v}}{Dt} = -\frac{1}{\rho} \overrightarrow{\nabla} P_m \cdot \overrightarrow{v} + \overrightarrow{a}_r \cdot \overrightarrow{v} \quad . \tag{22}$$

It is not difficult to see that our approximation results in total energy and momentum conservation. In particular:

1. Note from Eqs. (9) and (20) that the energy removed from the photon energy field via absorption is added to the material internal energy field, and that the energy added to the photon energy field via Planck emission is subtracted from material internal energy field.

2. Note from Eqs. (13), (14) and (21) that the momentum removed from the photon momentum field via absorption and scattering is added to the material momentum field.

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3. Note from Eqs. (9) and (22) that the change in the material kinetic energy due to radiation momentum deposition is subtracted from the radiation energy field.

Our corrected transport equation includes terms of order v^2/c^2. In particular:

$$
\begin{aligned}
C_0 &= (\sigma_a - \sigma_s)\, \overrightarrow{F}_\nu^* \cdot \frac{\overrightarrow{v}}{c} \\
&= (\sigma_a - \sigma_s) \left(\overrightarrow{F}_\nu \cdot \frac{\overrightarrow{v}}{c} - \frac{4}{3} c E_\nu \frac{v^2}{c^2} \right) \quad ,
\end{aligned}
\tag{23}
$$

It is significant to note that rigorous energy conservation is not obtained if all terms of order v^2/c^2 are neglected. For instance, as can be seen from Eqs. (7) and (13), the radiation acceleration contains only an $O(v/c)$ term:

$$
\overrightarrow{a}_r = \int_0^\infty \frac{\sigma_t}{\rho c} \left(\overrightarrow{F}_\nu - \frac{4}{3} c E_\nu \frac{\overrightarrow{v}}{c} \right) d\nu \quad ,
\tag{24}
$$

but when the radiation acceleration is used to calculate the rate of change of the material kinetic energy due to radiation momentum deposition, an $O(v^2/c^2)$ term arises:

$$
\overrightarrow{a}_r \cdot \overrightarrow{v} = \frac{\sigma_t}{\rho} \left(\overrightarrow{F}_\nu \cdot \frac{\overrightarrow{v}}{c} - \frac{4}{3} c E_\nu \frac{\overrightarrow{v}^2}{c^2} \right) \quad .
\tag{25}
$$

Thus if one neglects the $O(v^2/c^2)$ term in Eq. (25), one neglects to subtract from the radiation energy equation a small but non-zero fraction of the rate of change of the material kinetic energy due to radiation momentum deposition.

Mihalas and Klein [2] have given an Eulerian-frame transport formulation with material-motion corrections that is correct to $O(v/c)$. We now show that our radiation energy and

momentum equations are identical to their radiation energy and momentum equations to $O(v/c)$ in the equilibrium diffusion limit. The radiation energy equation of Mihalas and Klein is:

$$\frac{\partial E_r}{\partial t} + \overrightarrow{\nabla} \cdot \overrightarrow{F} + \int_0^\infty \sigma_a \left(cE_\nu - 2\overrightarrow{F}_\nu \cdot \frac{\overrightarrow{v}}{c} - 4\pi B \right) \, d\nu =$$

$$-\int_0^\infty \left(\sigma_t - \nu\frac{\partial \sigma_a}{\partial \nu} \right) \overrightarrow{F}_\nu \cdot \frac{\overrightarrow{v}}{c} \, d\nu \quad . \tag{26}$$

There are two differences between Eq. (26) and our radiation energy equation, Eq. (9):

1. Equation (26) contains $\overrightarrow{F}_\nu \cdot \frac{\overrightarrow{v}}{c}$ in various places rather than $\overrightarrow{F}_\nu^* \cdot \frac{\overrightarrow{v}}{c}$. However, $\overrightarrow{F}_\nu$ and $\overrightarrow{F}_\nu^*$ are identical to $O(v/c)$.

2. Equation (9) does not contain the term proportional to $\nu\frac{\partial \sigma_a}{\partial \nu}$ that appears in Eq. (26). This term does not appear in the equilibrium diffusion limit because the frequency dervative of the absorption cross-section must be "small" in a particular sense for the limit to exist.[3] Thus we find that Eqs. (9) and (26) agree to $O(v/c)$ in the equilibrium diffusion limit.

The radiation momentum equation of Mihalas and Klein is:

$$\frac{1}{c^2}\frac{\partial \overrightarrow{F}}{\partial t} + \overrightarrow{\nabla} \cdot \overrightarrow{P} + \int_0^\infty \frac{\sigma_t}{c} \overrightarrow{F}_\nu \, d\nu = \int_0^\infty (\sigma_a 4\pi B + \sigma_s cE_\nu) \frac{\overrightarrow{v}}{c} \, d\nu +$$

$$\int_0^\infty \left[\sigma_a + \sigma_s + \nu\frac{\partial \sigma_a}{\partial \nu} \right] \overrightarrow{P}_\nu \cdot \frac{\overrightarrow{v}}{c} \, d\nu \quad , \tag{27}$$

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where $\overrightarrow{\overrightarrow{P}}_\nu$ is the frequency-dependent radiation pressure:

$$\overrightarrow{\overrightarrow{P}}_\nu = \frac{1}{c} \int_{4\pi} \overrightarrow{\Omega}\,\overrightarrow{\Omega}\, I\, d\Omega \quad . \tag{28}$$

As noted previously, the term proportional to $\nu \frac{\partial \sigma_a}{\partial \nu}$ does not appear in the equilibrium diffusion limit. Furthermore, the following equivalences apply in the equilibrium diffusion limit,

$$4\pi B \to c E_\nu \quad , \tag{29}$$

and

$$\overrightarrow{\overrightarrow{P}}_\nu \to \frac{1}{3} E_\nu \quad . \tag{30}$$

Eliminating the term proportional to $\nu \frac{\partial \sigma_a}{\partial \nu}$ in Eq. (27) and substituting from Eqs. (29) and (30) into the material-motion terms in Eq. (27), we obtain Eq. (14). Thus Eqs. (14) and (27) are identical in the equilibrium diffusion limit.

In the equilibrium diffusion limit, the radiation energy and momentum equations represent a closed system for the radiation energy density and the total radiation flux. Since our radiation energy and momentum equations are correct to $O(v/c)$ in the equilibrium limit, it follows that our transport equation is also correct to $O(v/c)$ in that limit for angle/frequency-integrated intensity quantities.

Consider a simple problem consisting of an infinite medium in equilibrium at temperature T_0 moving at a constant velocity $\overrightarrow{v}_0$. Our corrected transport equation yields the

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following respective solutions for the angle/frequency-integrated radiation intensity and the radiation flux:

$$cE_r = acT_0^4 \quad , \tag{31}$$

where a is the radiation constant,

$$\overrightarrow{F} = -\frac{4}{3}cE_r\frac{\overrightarrow{v}_0}{c} \quad . \tag{32}$$

As expected, Eqs. (31) and (32) are correct to $O(v/c)$. However, the solutions for the fluid-frame angle-frequency integrated radiation intensity, the fluid frame radiation flux, and the radiation acceleration are exact:

$$cE_r^* = acT_0^4 \quad , \tag{33}$$

$$\overrightarrow{F}^* = 0 \quad , \tag{34}$$

$$\overrightarrow{a}_r = 0 \quad . \tag{35}$$

In closing we note that our corrected equation is exact in the streaming limit simply because the photons do not interact with the material in this limit. Thus there is no need to correct the cross sections. Indeed, if $\sigma_a = \sigma_s = 0$, our correction term is identically zero. This is an advantage of using the Eulerian-frame transfer equation rather than the fluid-frame equation. The fluid-frame equation retains complicated differential terms even when the cross sections are identically zero.

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References

[1] Dimitri Mihalas and Barbara Weibel Mihalas, *Foundations of Radiation Hydrodynamics*, Oxford University Press, New York, (1984).

[2] Dimitri Mihalas and Richard I. Klein, "On the Solution of the Time Dependent Inertial-Frame Equation of Radiative Transfer in Moving Media to $O(v/c)$," *J. Comp. Phys.*, **46**, 97 (1982).

[3] R. B. Lowrie, J. E. Morel, J. A. Hittinger, "The Coupling of Radiation and Hydrodynamics," accepted for publication in the *Astrophysical Journal*.